Traffic & Transportation

Approximation of Queues in Bike-Sharing Systems With Finite Docks

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Original Scientific Paper
Submitted: 22 May 2024
Accepted: 28 Aug 2024

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Publisher:
Faculty of Transport and Traffic Sciences, University of Zagreb

ABSTRACT

This paper presents a closed queuing network model to address bike queues in bike-sharing systems with finite docks. The model tackles issues of bike spillover and user attrition due to fully occupied docks and bike shortages at stations. The objective is to determine throughput rates and other performance metrics for these systems. To overcome computational challenges, we propose an approximation algorithm based on the developed model. Our analysis reveals intrinsic properties of bike-sharing systems with finite docks: (i) The effective system throughput rate increases with bike fleet size and eventually converges to a ceiling value. (ii) Adding more docks at stations can unnecessarily increase or even decrease the effective throughput rate. (iii) Under certain conditions, the system can reach a self-balancing state, avoiding bike surpluses or deficiencies at each station and maximising throughput. (iv) Users can successfully return bikes with a limited number of tries, provided there is at least one station on their route with a non-zero probability of having available docks. A small-scale artificial example and a case study demonstrate the accuracy and applicability of the approximation algorithm and the properties of the systems.

KEYWORDS

finite-docked bike-sharing system; throughput rate; approximation algorithm; closed queueing network.

1. INTRODUCTION

Docked bike-sharing systems (DBSSs) provide shared micro-mobility services that promote cycling, urban mobility and public transportation usage. By April 2022, over 1,700 DBSSs operated more than 8.93 million bikes worldwide [1]. However, finite numbers of bikes and docks can lead to stations lacking available bikes for rent or idle docks for returns, affecting service quality [2], efficiency [3] and user satisfaction [4]. Understanding the relationship between DBSS efficiency and its deployment, including the number of docks, stations and bike fleet sizes, is crucial for effective planning and operations.

Over the past decade, extensive research on DBSS planning and operation has been conducted at strategic, tactical and operational levels. Strategic-level planning involves long-term infrastructure decisions, such as bikeway network design [5–7], station network design [8–10] and fleet sizing [2, 11]. Tactical-level operations address medium-term decisions, including bike inventory levels [12–13] and user demand management [2, 14–17]. Operational-level operations focus on short-term bike relocation to respond to dynamic user demands [15, 18–21]. For a comprehensive review of DBSS literature, see Shui and Szeto [22]. Previous studies have significantly reduced costs, improved service performance and increased profits through optimising stations, bikeways, bike relocation and user incentives.

DBSS operation has three distinct characteristics. First, DBSSs have a limited number of docks per station, which can cause bike to spill over from a station. Consequently, users may need to divert to nearby stations to return their bikes. To emphasise the limitation on the number of docks at each station, we refer to these systems as finite docked bike sharing systems (FDBSSs). Second, there is uncertainty associated with user demand and cycling time. User arrivals at stations and cycling times between stations are inherently random due to user independence and varying cycling speeds. Third, the bike fleet size in a DBSS is approximately fixed over short periods (e.g. one day or half a week) because all bikes must circulate among stations within the system, forming a closed queuing network. Considering these characteristics, treating a finite-docked bike-sharing system (FDBSS) as a closed queuing network under uncertainties appears to be a promising approach.

Several models have been proposed to analyse FDBSSs within the framework of closed queueing network. George and Xia [23] first used a closed queueing network to optimise fleet size, assuming an infinite number of docks at stations. Fricker and Gast [2] developed a model to optimise bike fleet size, aiming to minimise the number of fully occupied or vacant docks, assuming users could always return bikes at available stations. Li et al. [24] proposed a closed queueing network model considering multi-class users by treating bikes as virtual consumers and stations and bikeways as virtual nodes. Due to the exponential growth of virtual nodes, calculating the steady-state probabilities of bike distributions is challenging [25]. Celebi et al. [9] presented a decision model to minimise unsatisfied bike rental and return demand by optimising station location and dock allocation, without considering user origin-destination distributions. Vishkaei et al. [26] extended George and Xia's model [23] to optimise fleet size, dock allocation and bike flow, while minimising the average number of lost users, taking into account station capacity limitations. Despite these efforts, systematic analysis of the FDBSS efficiency with service failures remains limited. Furthermore, analysing the FDBSS by using the closed queueing network theory requires computing the steady-state probabilities of bike distributions, which grow exponentially with the number of bikes, stations and docks, significantly increasing computational complexity.

Despite significant advancements in the FDBSS research, several gaps persist. First, the existing FDBSS planning and operation models heavily rely on optimisation, necessitating dynamic and precise user demand data. However, during the planning stage, only approximate daily or hourly user demands are available. Current studies, particularly data-driven ones, struggle to assess system service capabilities and their relationships with bike and dock quantities under limited user demand and riding data conditions. Second, bike rental and return failures impact the quality and efficiency of the FDBSS services. Methods for diagnosing which stations will experience bike surpluses or deficiencies under uncertainties have not been developed. Third, previous closed queuing network models for FDBSSs often require computing normalisation constants for exact steady-state probabilities of bike distributions, a process that is time-consuming and limits their applicability to large-scale FDBSSs. Therefore, an approximation algorithm is needed to estimate bike queues, i.e. the number of bikes at stations or on bikeways.

To address the existing gaps in the FDBSS research, we model bike rental and return processes at an aggregate level, focusing on dock limitations and macroscopic performance evaluations. Our study aims to answer several fundamental questions: How can the FDBSS service efficiency be effectively measured? Can service efficiency be improved by increasing the bike fleet size and the number of docks? Can bike surpluses or deficiencies be avoided at each station without bike relocation? Addressing these questions necessitates an approximation algorithm to estimate bike queues at stations and reveal the inherent operational properties of FDBSSs.

Our contributions are threefold. First, we develop a closed queuing network formulation for FDBSS, thus deriving an approximation algorithm to determine performance metrics, including throughput rates, expected bike quantities on bikeways and at stations, bike dwell times and the probability of rental and return failures at stations. Second, we explore the inherent characteristics of the FDBSS, focusing on the impacts of bike fleet size and dock allocation on throughput rates and identify the critical conditions required for achieving FDBSS self-balancing. Despite its importance for the design and planning of bike-sharing systems, the FDBSS characteristics have not been fully explored when docks are limited and demands are uncertain. Third, we introduce a diagnostic method to predict which stations may experience bike surpluses or deficiencies, even under uncertain user demands and variable riding speeds. Our model is less data-intensive and suitable for evaluating potential FDBSS design schemes during the planning stage. To the best of our knowledge, such methods have not been explored in previous studies.

The structure of this paper is as follows: Section 2 formulates the FDBSS and develops a throughput rate approximation algorithm. Section 3 examines the operational properties of the FDBSS and introduces a diagnostic method for forecasting bike surpluses and deficiencies at each station. Section 4 provides a small-scale example and a case study to illustrate the model. Section 5 concludes the paper. Due to space constraints, proofs of the propositions, corollaries and FDBSS properties are omitted. Interested readers may request these from the authors.

2. MODEL FORMULATION

2.1 Problem description

Consider a FDBSS with m stations and a fleet of n bikes, where station j, $j=1,2,...,m$ has B_j docks. Suppose users departing from station i, $i=1,2,...,m$ may ride to station j with probability p_{ij}, $p_{ij} \geq 0, \sum_{j=1}^{m} p_{ij} = 1, i, j=1,2,...,m$. We call $P=[p_{ij}], i, j=1,2,...,m$ the routing matrix. Users can return bikes to their destination station j if there are docks available; if all docks at station j are full, they may ride to next station, say, s, with probability η_{js} to return bikes. We will refer to $\eta=[\eta_{js}]$, $j, s=1,2,...,m$ as the overflow routing matrix. If station s is still fully occupied, they continue riding to other stations until they can successfully return bikes. In this process, those users who cannot return bikes at their destination stations are served inefficiently.

Depending on whether bikes can be rented and returned successfully, a station may have four types of bike arrivals and departures. The first type is the arrival bikes, which are voluntarily returned to the station that is their intended destination. The second type is also arrival bikes, which cannot be returned to other fully occupied stations and are involuntarily forced to be returned at the station. The third type is the departure bikes, which are rented at and voluntarily leave the station. The fourth type is also the departure bikes, which cannot be returned and are involuntarily forced to leave the fully occupied station. In the long run, based on the principle of bike conservation, the number of bikes arriving at a station should equal the number of bikes departing. Thus, at a station, the total of the first and second arrival bikes is equal to the total of the third and fourth departure bikes. This is a fundamental fact for model development. Note that the second and fourth types of bike flow are regarded as inefficient as those users cannot directly reach their destination stations. The bike classification allows for a more accurate evaluation of the FDBSS service performance.

A key metric for measuring FDBSS efficiency is the user service completion rate, which is equivalent to the throughput rate of a queueing system [25, 27]. We aim to formulate the nominal and effective system throughput rates under a closed queuing network framework, where the former quantifies the third and fourth bike departure flows, while the latter is only the third one.

In this paper, we make the following assumptions to clarify the premises for the model formulation:

A1. The period concerned is the morning or evening rush hours because these periods may have the most serious bike surplus and deficiency [2]. We assume that users arrive at station j with a Poisson rate λ_j, $j=1,2,...,m$, during rush hours. Such an assumption has been wildly adopted in previous related studies, e.g. Çelebi et al. [9], George and Xia [23].

A2. We assume that the users between stations i and j take exponential cycle times with rate τ_{ij}, $i, j=1,2,...,m$ due to the heterogeneities in cycling speed and call $\tau=[\tau_{ij}], i, j=1,2,...,m$ the ride time matrix. This assumption was also used in previous related studies [23, 24, 26].

A3. We assume that arriving users will leave the FDBSS if there are no available bikes at the stations and that if the docks at a station are fully occupied, users will ride to the next station to return bikes; if the station is also full, users will continue riding to another station until they can successfully return bikes [9].

2.2 Closed queueing network

In this section, we first present a closed queueing network formulation for the FDBSS under uncertainty. We then prove that a user arrival flow at an origin station can partly transform into a Poisson bike arrival flow at a destination station, demonstrating that the FDBSS exhibits the M ⇒M property (M implies Markov). The M ⇒M property says that in a closed queueing network, a station behave as if it were separate service system [25]. This characteristic means that any station can be examined in isolation from the rest of the FDBSS. Next, we propose a formulation for the FDBSS network to develop a counterpart version of the M⇒M property proposition. We then model the behaviour of a station as a single-server system by formulating the bike arrival and departure flows at an individual station. Note that the user arrivals hereinafter all refer to users wanting to rent bikes at stations.

Network-based formulation

According to George and Xia [23] and Li et al. [24], treating an FDBSS as a closed queuing network allows user arrivals to be interpreted as virtual service for available bikes and bike arrivals as job arrivals. Following this approach, we model user inter-arrival times as the service times for each bike and job arrival times as bike arrival times at each station. This method enables us to analyse bike traffic in an FDBSS using closed queuing network theory.

For any bike in FDBSS, under steady state equilibrium, the time proportion of the bike dwelling at station i, π_i, i.e. the ratio of a bike's dwelling time at the station to its overall dwelling times at stations, is equal to the sum of the products of the time proportion at station i and the routing probabilities from station i to station j, p_{ij}. Then, π_j is the unique positive solution for the equations

$$\begin{cases} \sum_{i=1}^{m} \pi_i p_{ij} = \pi_j \\ \sum_{j=1}^{m} \pi_j = 1 \end{cases} \tag{1}$$

The nominal throughput rate at station j, denoted as $a_j(n)$, is the sum of the products of the flow rates of bike departures from total source stations (including station j itself) and the routing probability to station j. Specially, $a_j(n)$ is calculated as $a_j(n)$ is, where $a_j(n)$ is the flow rate of bike departures from station and p_{ij} routing probability from station i to station. Thus, we have the following FDBSS traffic equations:

$$a_j(n) = \sum_{i=1}^{m} a_i(n) p_{ij} \tag{2}$$

From *Equations 1* and *2* it follows:

$$\begin{cases} a_j(n) = a(n)\pi_j \\ a(n) = \sum_{i=1}^{m} a_i(n) \end{cases} \tag{3}$$

The following proposition can provide the foundation that the FDBSS is equivalent to a separable closed queueing network, where the stations behave as if they were in isolation from the rest of the system.

Proposition 1 (Property of M⇒M). Given each station with finite docks, if users arriving at each station to rent bikes follow a Poisson process, then, despite bikes potentially spilling over due to fully occupied docks, the arrivals of bikes at each station also follow a Poisson process.

Proposition 1 guarantees that a user arrival flow at each origin station can partly transform into a Poisson bike arrival flow at each destination station, thus allowing any station in FDBSSs to behave as if it were a single node independent of the rest of the system. This characteristic enables us to decompose the analysis of queues in the FDBSS into that of a single station.

Station-based formulation

Proposition 1 allows us to analyse the queueing process at a station in detail instead of the overall FDBSS. By assumption A3, there is no user queue if the station has vacant docks. Therefore, station j can be treated as an M/M/1/B_j queue system. In this context, the inter-arrival times of bikes at the station follow an exponential distribution and the service times, which correspond to the intervals between arrivals of users wanting to pick up a bike, also follow an exponential distribution. The '1' denotes a single server, which is the arrival flow of users who want to rent bikes at the station; and 'B_j' denotes the station's capacity, representing the number of bikes that can be parked at station j. In our model, the queue represents bikes parked at the station, waiting to be picked up by users. When a user arrives and finds a vacant bike, they immediately take a bike, reflecting the service mechanism of the system. If all docks are occupied, arriving bikes will divert to the next station to return bikes, which aligns with the M/M/1/B_j queue system's capacity constraints. According to the state transitions of the bike rental and return at the station, as shown in *Figure 1*, letting k bikes at station j denotes the state of the station, the rates of entering and leaving the state of k bikes at station j, $a_j(k)$ and $\lambda_j(k)$ can be written as follows:

$$a_j(k) = \begin{cases} a_j(n), \ k=0,1,\cdots,B_j\text{-}1 \\ 0, \ k=B_j, B_j+1, \cdots \end{cases} \quad (4)$$

$$\lambda_j(k) = \lambda_j, \quad k=1,\cdots,B_j \quad (5)$$

where $a_j(n)$, $j=1,2,...,m$ also denotes the average bike arrival rate at station j in the FDBSS with n bikes and can be solved from the traffic equations in *Equation 2*.

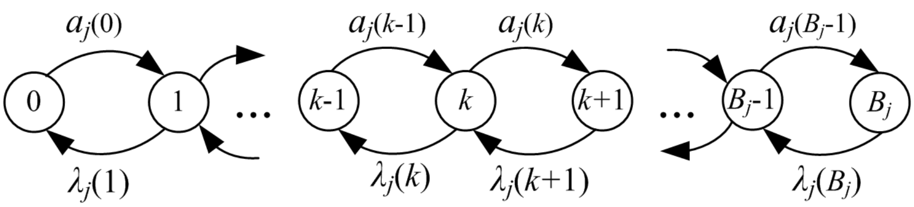


Figure 1 – Transition states for bike rental and return at a station

Under steady-state conditions, the rate of entering the state of k bikes equals the rate of leaving the state at the station. Based on the fact that the state of k bikes can be changed to the states of $k-1$ and $k+1$ bikes only, we have the following balance equations:

$$a_j P(X_j(t)=k-1) + \lambda_j P(X_j(t)=k+1) = (a_j+\lambda_j) P(X_j(t)=k), \quad k=1,2,\cdots,B_j-1 \quad (6)$$

Owing to the fact $\sum_{k=0}^{B_j} P\{X_j(t)=k\}=1$ and letting $\rho_j = a_j/\lambda_j$, we can solve the probabilities of vacant and fully occupied docks at station j as follows:

$$P(X_j=0) = \begin{cases} \dfrac{1-\rho_j}{1-\rho_j^{B_j+1}}, & \rho_j \neq 1 \\ \dfrac{1}{B_j+1}, & \rho_j = 1 \end{cases} \quad (7)$$

$$P(X_j=B_j) = \begin{cases} \dfrac{(1-\rho_j)\rho_j^{B_j}}{1-\rho_j^{B_j+1}}, & \rho_j \neq 1 \\ \dfrac{1}{B_j+1}, & \rho_j = 1 \end{cases} \quad (8)$$

By **assumption A3**, the presence of vacant and fully occupied docks at stations may affect bike arrival rates and thus change the elements in the routing matrix P. Now, we compute the actual routing matrix in the FDBSS. Specifically, the average bike arrival rate from station i to station j includes voluntary and involuntary arrival rates, $\sum_i \lambda_i (1-P(X_i=0)) p_{ij}$, $\sum_i a_i P(X_i=B_i)\eta_{ij}$, respectively. Thus, the actual routing probability from station i to station j is equal to the proportion of bike flow from station i to station j to total bike arrival flows at station j. Then, we have the following:

$$\bar{p}_{ij} = \frac{\lambda_i \left(1 - P(X_i = 0)\right) p_{ij} + a_i P(X_i = B_i)\eta_{ij}}{\sum_{i=1}^{m} \left\{ \lambda_i \left(1 - P(X_i = 0)\right) p_{ij} + a_i P(X_i = B_i)\eta_{ij} \right\}} = \frac{\lambda_i \left(\rho_i - \rho_i^{B_i+1}\right) p_{ij} + a_i \left(1 - \rho_i\right) \rho_i^{B_i} \eta_{ij}}{\sum_{i=1}^{m} \left\{ \lambda_i \left(\rho_i - \rho_i^{B_i+1}\right) p_{ij} + a_i \left(1 - \rho_i\right) \rho_i^{B_i} \eta_{ij} \right\}} \tag{9}$$

where if station j is the next return station for station i, then $\eta_{ij} > 0$; otherwise $\eta_{ij} = 0$. Here, station j is the destination node for station i; station i, $i = 1, 2, ..., m$ is the source nodes for station j. Indeed, stations i and j can be both the source and destination nodes for each other, hence *Equations 7–9* can be applied to all stations.

2.3 Approximation algorithm

This section aims to derive an approximation algorithm for computing system throughput rate and other service performance metrics. Recall that calculating the exact throughput rate requires solving the steady-state probabilities of bike distributions in a closed queueing network [25], which is very time-consuming even for those with small numbers of stations and jobs [25, 27–28]. For a deeper understanding of the computational challenges involved, interested readers can refer to the detailed analyses by Li et al [24] and Bolch et al. [25].

To this end, we employ the Mean Value Analysis [28] to derive an approximation algorithm for system throughput rates. Note that bikes do not leave the station in the order they arrive; that is, stations do not operate according to the first-come, first-served (FIFO) rule. According to the PASTA (Poisson Arrivals See Time Averages) property of the Arrival Theorem [27], the distribution of bikes for FDBSS with n bikes, as observed by a bike arriving at a station, is equivalent to the stationary distribution for FDBSS with $n-1$ bikes. This property holds under more general conditions and is not limited to FIFO service. In our analysis, the key requirement is that the bike arrivals are Poisson and the system is in a steady state. Under these conditions, the arriving bike observes the system as it would be seen by an external observer at a random point in time, irrespective of the service discipline employed. Thus, according to the PASTA property of the Arrival Theorem, the expected dwell time at station j for n bikes is equal to the sum of the expected dwell time of that bike and the total expected dwell times of all bikes except that bike, which is as follows:

$$E(T_j(n)) = \frac{1 + E(N_j(n-1))}{\lambda_j} \tag{10}$$

where $N_j(n)$ and $T_j(n)$ represent the bike amount (i.e. queue length) and dwell time at station j in the FDBSS with n bikes. A bike's circulation time in the FDBSS includes its total dwell times at stations and cycling times on bikeways, i.e. $\sum_{j=1}^{m} \pi_j E(T_j(k)) + \sum_{i=1}^{m} \sum_{j=1}^{m} \pi_i \bar{p}_{ij} \tau_{ij}$. The system throughput rate can be treated as the frequency at which total bikes traverse total bikeways and stations, from which it follows:

$$a(n) = n \Big/ \left[\sum_{j=1}^{m} \pi_j E(T_j(k)) + \sum_{i=1}^{m} \sum_{j=1}^{m} \pi_i \bar{p}_{ij} \tau_{ij} \right] \tag{11}$$

The effective throughput rate for station j corresponds to the minimum of the expected bike arrival rate and the user arrival rate at that station. Thus, we have the following:

$$TH_j(n) = \min(a_j, \lambda_j) \tag{12}$$

where $a_j = a(n)\pi_j$ denotes expected bike arrival rate at station j in the FDBSS with n bikes.

By Little's formula, the expected number of bikes at station j can be written as follows:

$$E(N_j(n)) = \min(TH_j(n) E(T_j(n)), B_j) \tag{13}$$

Using *Equations 1–3, 7–13*,we can design an approximation algorithm to calculate the system throughput rate. The approximation algorithm is outlined in the pseudo-code below and its solutions can be verified using simulation. This approximation algorithm is attractive because it does not require calculating joint probabilities for all possible combinations of bike numbers at all stations, making it efficient for large-scale FDBSSs.

Algorithm 1 – Approximate algorithm

Input: Users arrive at station j with a Poisson rate λ_j, $j=1,2,...,m$, bike fleet size n, number of docks at each station B_j, $j=1, 2, ..., m$, average bike cycle time τ_{ij} between stations i and j and routing matrix $P=[p_{ij}]$, $i, j=1, 2, ..., m$.

Output: System and station effective throughput rates TH and TH_j, average bike inventory $\mathrm{E}(N_j)$, dwell times $\mathrm{E}(T_j)$ and probabilities of surplus and deficiency $\mathrm{P}(X_j=0)$ and $\mathrm{P}(X_j=B_j)$ at each station.

Initialise: Solve *Equation 1* to obtain limiting probabilities;

Step 1. For $j=1,2, ..., m$

Let $E(N_j(0))=0$ and $a(0)=0$;

Step 2. For $k=1,\ldots,n$, do

Compute the expected dwell time at station *i* for *n* bikes from *Equation 10*;

If $k \le \min(B_j)$ then

$a(k)=k\big/\left[\sum_{j=1}^{m}\pi_j E(T_j(k))+\sum_{i=1}^{m}\sum_{j=1}^{m}\pi_i p_{ij}\tau_{ij}\right]$;

$a_j(k)=a(k)\pi_j$;

Otherwise

Solve the actual routing probability from *Equations 10, 11 and 12*;

Solve *Equation 1*;

$a(k)=k\big/(\sum_{j=1}^{m}\pi_j E(T_j(k))+\sum_{i=1}^{m}\sum_{j=1}^{m}\pi_i p_{ij}\tau_{ij})$;

$a_j(k)=a(k)\pi_j$;

End if

$TH_j(k)=\min(a_j(k),\lambda_j)$;

$TH(k)=\sum_{i=1}^{m}TH_i(k)$;

$E(N_j(k))=\min(TH_j(k)E(T_j(k)),B_j)$;

End

End

This algorithm is very easy to be coded and can even be carried out manually with the aid of an electronic calculator for small values of n, m and B_j.

Based on the system throughput rates, we can evaluate various system performance metrics listed in *Table 1*. These metrics provide insights into the FDBSS service levels from different perspectives. For example, the effective throughput rate can identify stations with low availability, while the ineffective throughput rate (nominal minus effective) can highlight stations with a high rate of failed bike returns, guiding potential FDBSS improvements. In addition, by analysing the average bike inventory, dwell times and probability of surplus and deficiency at each station, the operators may make informed decisions on the FDBSS deployments and bike relocations.

Table 1 – FDBSS service performance metrics

Indicator	Description	Formula
Effective system throughput rate	Users successfully rent bikes per unit time	$TH(n)=\sum_{i=1}^{m} TH_i(n)$
Effective station throughput rate	Users successfully renting bikes at station j per unit time	$TH_j(n)=\min\left(\frac{n\pi_j}{\sum_{i=1}^{m}\pi_j E(T_j(n))+\sum_{i=1}^{m}\sum_{j=1}^{m}\pi_i \bar{p}_{ij}\tau_{ij}},\lambda_j\right)$
Nominal system throughput rate	Users riding bikes from all stations per unit time	$a(n)=n/(\sum_{i=1}^{m}\pi_i E(T_j(n))+\sum_{i=1}^{m}\sum_{j=1}^{m}\pi_i \bar{p}_{ij}\tau_{ij})$
Ineffective system throughput rate	Users failing to return bikes per unit time	$a(n)-TH(n)$
Expected bike inventory at station	Expected number of bikes at station j	$E(N_j(n))=\min\left(TH_j(n)E(T_j(n)),B_j\right)$
Expected bike dwell Time at station	Average bike stay duration at station j	$E(T_j(n))=\left(E(N_j(n-1))+1\right)/\lambda_j$
Probability of bike deficiency at station	The probability that the number of bikes at station j is less than $\varphi_1 B_j$	$P(X_j<\lfloor\varphi_1 B_j\rfloor)$
Probability of bike surplus at station	The probability that the number of bikes at station j is greater than $\varphi_2 B_j$	$P(X_j>\lfloor\varphi_2 B_j\rfloor)$

Note: $0\le\varphi_1\le 1$ and $0\le\varphi_2\le 1$ are the upper and lower percentages of bikes inventories relative to docks, which are usually set by the FDBSS operators to initiate the bike relocations when bike inventories are out of the upper and lower thresholds.

3. FDBSS PROPERTIES

The FDBSS differs from traditional queuing systems in that its stations do not have dedicated servers for bike rental and return operations. Consequently, some properties of classic closed queuing networks may not always apply to FDBSS. In this section, we explore the inherent properties of FDBSS to understand how various resources, such as the number of bikes and docks, affect service performance metrics. We begin by presenting the main property of the BSS throughput rate.

Property 1. The FDBSS throughput rate does not decrease as the bike fleet size increases, given the dock allocation.

Section 4 will show that the effective system throughput rate initially increases with the bike fleet size and eventually converges to a ceiling value. Thus, increasing the fleet size beyond a certain value – optimal bike fleet size – cannot always raise the throughput rate. We are now prepared to define the optimal bike fleet size.

Definition 1 (Optimal fleet size). A bike fleet size is said to be optimal if $n_{\text{OPT}}=\max\{k\,|\,TH(k)-TH(k-1)>0 \text{ and } TH(k+1)-TH(k)=0\},\ k\in Z^{+}$.

Remark. The approximation algorithm can easily find the optimal bike fleet size by iterating k from 1 to fleet size n and calculating $TH(k+1)$, $TH(k)$ and $TH(k-1)$. When the fleet size exceeds the optimal level, the additional bikes not only fail to increase the FDBSS service capacity but also reduce service quality, making them uneconomical. This will be observed in Section 4.

The fleet size influences both bike dwell times at stations and bike utilisation. The following property provides further insight.

Property 2. Increasing the bike fleet size in BSS cannot decrease the expected dwell time of bikes at station j, i.e. $E(T_j(n+1))\ge E(T_j(n)),\ j=1,2,...,m$.

Property 2 indicates that releasing a larger fleet size can increase the expected bike dwell times at stations, leading to longer bike circulation times in the FDBSS. While **Property 3** demonstrates that longer bike circulation times can improve effective bike utilisation.

Property 3. The effective bike utilisation, $\delta(n)=\sum_{i=1}^{m}\sum_{j=1}^{m}TH_i(n)\cdot E(T_{ij}(n))/n$, decreases with bike fleet size n and increases with longer expected circulation time, $\sum_{i=1}^{m}\pi_i E(T_i(k))+\sum_{i=1}^{m}\sum_{j=1}^{m}\pi_i\bar{p}_{ij}\tau_{ij}$, given dock allocation and number of stations.

The unanswered question is whether the initial bike allocation for the FDBSS stations affects the throughput rates. The following property provides the answer.

Property 4. Given the dock allocation and bike fleet size, the initial bike allocation for stations cannot affect the FDBSS and station throughput rates.

Remark. Since bike rental and return services are Markov processes, the FDBSS can reach equilibrium states after a long time of operation. Therefore, the throughput rates do not depend on the initial bike allocation due to FDBSS's memoryless property.

Identifying bike surplus or deficiency is crucial, as it can lead to a negative user experience and prompt users to switch to alternative transport modes. To clarify the conditions causing bike surplus and deficiency at stations, we begin with the following definition.

Definition 2 (Bike surplus or deficiency at stations). (i) Station j is said to be bike deficient if the probability of the station having fewer than $\varphi_1 B_j$ bikes is greater than ω_1, i.e. $P(X_i<\lfloor\varphi_1 B_j\rfloor)>\omega_1$; (ii) Station j is said to be bike surplus if the probability of the station having more than $\varphi_2 B_j$ bikes is greater than ω_2, i.e. $P(X_j>\lfloor\varphi_2 B_j\rfloor)>\omega_2$, where $0<\omega_1<1$ and $0<\omega_2<1$ are the thresholds of probabilities that bikes inventories are less than or exceed their respective upper and lower limits, as usually determined by the FDBSS operators; $\lfloor\cdot\rfloor$ denotes a floor integer.

The following proposition and its corollary provide a diagnostic method to predict which stations are bike surplus or deficient.

Proposition 2.

(i) Station j is bike surplus if the arrival rates of bikes and users, a_j and λ_j, satisfy the following:

$$\begin{cases}\dfrac{a_j^{\lfloor\varphi_2 B_j\rfloor+1}\cdot\lambda_j^{B_j-\lfloor\varphi_2 B_j\rfloor}-a_j^{B_j+1}}{\lambda_j^{B_j+1}-a_j^{B_j+1}}>\omega_2, & \dfrac{a_j}{\lambda_j}\neq 1\\ \dfrac{B_j-\lfloor\varphi_2 B_j\rfloor}{B_j+1}>\omega_2, & \dfrac{a_j}{\lambda_j}=1\end{cases} \tag{14}$$

(ii) Station j is bike deficient if bike and user arrival rates, a_j and λ_j, satisfy the following:

$$\begin{cases}\dfrac{\lambda_j^{B_j+1}-a_j^{\lfloor\varphi_1 B_j\rfloor}\cdot\lambda_j^{B_j-\lfloor\varphi_1 B_j\rfloor+1}}{\lambda_j^{B_j+1}-a_j^{B_j+1}}>\omega_1, & \dfrac{a_j}{\lambda_j}\neq 1\\ \dfrac{\lfloor\varphi_1 B_j\rfloor}{B_j+1}>\omega_1, & \dfrac{a_j}{\lambda_j}=1\end{cases} \tag{15}$$

(iii) Station j is bike balancing if bike and user arrival rates, a_j and λ_j, satisfy the following:

$$\begin{cases}\dfrac{\lambda_i^{B_j+1}-a_i^{\lfloor\varphi_1 B_j\rfloor}\cdot\lambda_i^{B_j-\lfloor\varphi_1 B_j\rfloor+1}}{\lambda_j^{B_j+1}-a_j^{B_j+1}}\le\omega_1 \text{ and } \dfrac{a_j^{\lfloor\varphi_2 B_j\rfloor+1}\cdot\lambda_j^{B_j-\lfloor\varphi_2 B_j\rfloor}-a_j^{B_j+1}}{\lambda_j^{B_j+1}-a_j^{B_j+1}}\le\omega_2, & \dfrac{a_j}{\lambda_j}\neq 1\\ \dfrac{\lfloor\varphi_1 B_j\rfloor}{B_j+1}\le\omega_1 \text{ and } \dfrac{B_j-\lfloor\varphi_2 B_j\rfloor}{B_j+1}\le\omega_2, & \dfrac{a_j}{\lambda_j}=1\end{cases} \tag{16}$$

If the probability of station j having zero bikes is greater than ω, i.e. $P(X_j=0)>\omega$, then station j is said to be vacant. If the probability of station j having B_i bikes is greater than ω, i.e., $P(X_j=B_j)>\omega$, then station j is said to be full. Letting $\varphi_1 B_j=1$, $\varphi_2 B_j=B_j-1$, $\omega_1=\omega_2=\omega$ can immediately obtain the following **Corollary 1** of **Proposition 2**.

Corollary 1.
(i) Station j is full if bike and user arrival rates, a_j and λ_j, satisfy the following:

$$\begin{cases} \dfrac{a_j^{B_j}}{\sum_{k=0}^{B_j} \lambda_j^{B_j-k} a_j^k} > \omega, & \dfrac{a_j}{\lambda_j} \neq 1 \\ \dfrac{1}{B_j+1} > \omega, & \dfrac{a_j}{\lambda_j} = 1 \end{cases} \tag{17}$$

(ii) Station j is vacant if bike and user arrival rates, a_j and λ_j, satisfy the following:

$$\begin{cases} \dfrac{\lambda_j^{B_j}}{\sum_{k=0}^{B_j} \lambda_j^{B_j-k} a_j^k} > \omega, & \dfrac{a_j}{\lambda_j} \neq 1 \\ \dfrac{1}{B_j+1} > \omega, & \dfrac{a_j}{\lambda_j} = 1 \end{cases} \tag{18}$$

(iii) Station j is bike-balanced if bike and user arrival rates, a_j and λ_j, satisfy the following:

$$\begin{cases} \dfrac{\lambda_j^{B_j}}{\sum_{k=0}^{B_j} \lambda_j^{B_j-k} a_j^k} \leq \omega \text{ and } \dfrac{\alpha_j^{B_j}}{\sum_{k=0}^{B_j} \lambda_j^{B_j-k} a_j^k} \leq \omega, & \dfrac{a_j}{\lambda_j} \neq 1 \\ \dfrac{1}{B_j+1} \leq \omega, & \dfrac{a_j}{\lambda_j} = 1 \end{cases} \tag{19}$$

Using **Proposition 2** and **Corollary 1**, we can identify stations with bike surpluses or deficiencies, thereby determining the origin-destination (OD) distribution needed for scheduling bike relocation trucks. **Proposition 3** establishes the simple self-balance condition that allows FDBSS to operate at a maximal throughput rate without the need of bike relocation. Let $\mathbf{P}^{\mathrm{T}}$ denotes the transpose of matrix P, we have the following:
Proposition 3. (i) The necessary and sufficient condition for the FDBSS throughput rates to be maximised is $\left(\mathbf{I}-\mathbf{P}^{\mathrm{T}}\right)\boldsymbol{\lambda}=\mathbf{0}$; (ii) if a BSS satisfies this condition, each station is bike-balancing.

Remark. The condition $\left(\mathbf{I}-\mathbf{P}^{\mathrm{T}}\right)\boldsymbol{\lambda}=\mathbf{0}$ is challenging to satisfy in the real-world FDBSS scenarios, as even minor perturbations in the routing matrix P and user arrival rate vector λ can disrupt it. Moreover, incentivising users to alter their original travel intentions to meet this condition is difficult, as regulating the extent of user incentives precisely is challenging.

According to **Assumption A3**, docks can be fully occupied at certain stations, even if the probability of having non-full docks is less than one. Consequently, users may be unable to return bikes at these stations and must seek other stations. This scenario raises the possibility of users repeatedly attempting to return bikes at stations with temporarily full docks, but consistently failing. To address this issue, we define a single bike return, followed by **Proposition 4**.

Definition 3. A user attempting to return a bike to a station, regardless of success, is considered one return attempt.

Proposition 4. In an FDBSS, users can successfully return bikes a limited number of times, provided there is at least one station along their route with a probability of fully occupied docks less than 1.

Proposition 4 reveals two intriguing bike return patterns. We next discuss the first pattern. Consider users who are only aware of stations 1–4, as shown in *Figure 2* and can only return bikes to these stations. If the probability of fully occupied docks at each of these four stations is 1, users must continuously circulate among these stations to return bikes, as no bikes can be returned. This pattern is unfavourable due to its inefficiency.

Next, we discuss the second pattern. Here, users are aware of stations 5–10, also shown in *Figure 2*, and can return bikes only to these stations. Suppose station 10 consistently has available docks due to its high user arrival rate, while stations 5–9 always have fully occupied docks (i.e. the probability of fully occupied docks at these five stations is 1. In this pattern, all users, regardless of their initial station, can only successfully return bikes at station 10, resembling a black hole where all bike returns are concentrated.

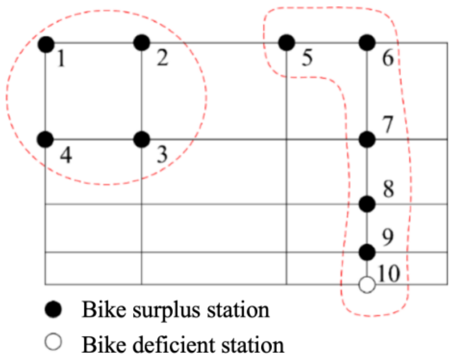


Figure 2 – A FDBSS with bike surplus and deficient stations

Proposition 4 shows that if the probability of fully occupied docks at each station is less than 1, the expected number of times a user returns a bike is limited. In particular, when the FDBSS and user arrival are uniform, i.e. have identical λ_j, B_j, p_{ij}, η_{ij} and τ_{ij}, and hence identical probabilities of fully occupied docks at stations, we have the following corollary.

Corollary 2. For the uniform FDBSS, the expected number of tries a user returns a bike is $1/(1-p)$, where $0 < p < 1$ is the probabilities of fully occupied docks at each station.

A uniform FDBSS implies that each station has an identical probability of fully occupied docks. Users attempting to return bikes may encounter the identical probability of a successful return at any station and hence may experience the same expected number of times returning bikes.

4. NUMERICAL EXPERIMENTS

This section presents two examples to validate the proposed approximation algorithm and examine the operational properties of the FDBSS. In Section 4.1, a small-scale artificial example is used to illustrate the performance of the algorithm, the properties of the FDBSS and the diagnostic approach for identifying stations with bike surpluses and deficiencies. Section 4.2 applies the developed approximation algorithm to a real-world case to demonstrate its practical application. To test the validity of the approximate algorithm, we developed a Markov simulation program to simulate the arrivals, departures and movements of users and bikes within the network.

4.1 Experiments on a small artificial BSS

Figure 3 shows a small FDBSS with four stations, as well as the average cycle times, routing matrix P and overflow routing matrix η. The number of docks at all four stations is identical, denoted as B. The average user arrival rate at each station is set at 10 pax/min. The initial numbers of bikes and docks at each station both range synchronously from 1 to 100.

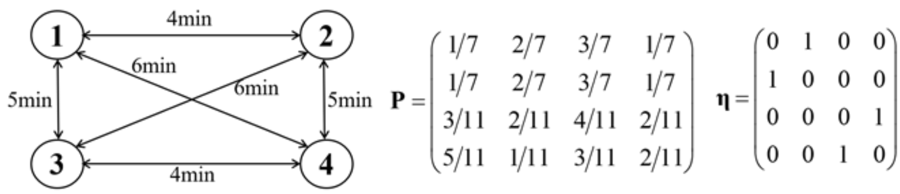


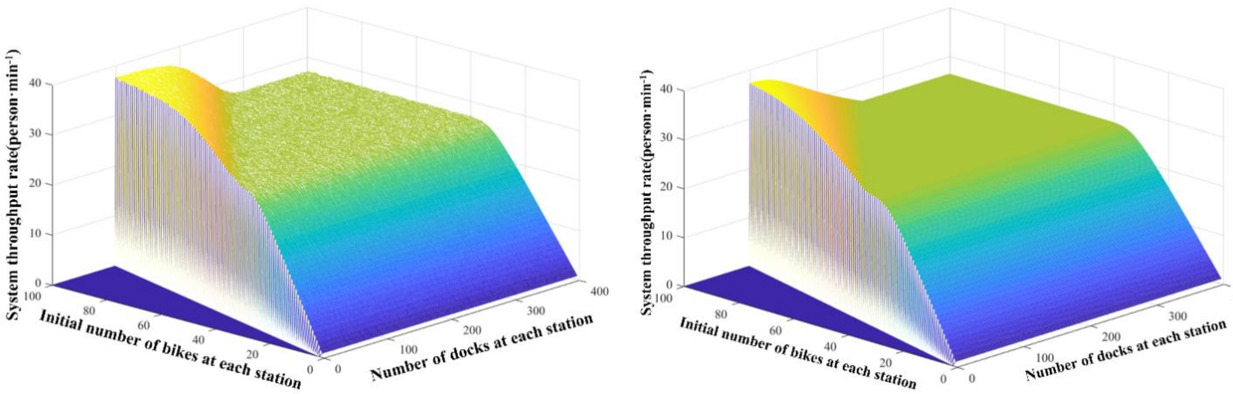
Figure 3 – Small artificial BSS

Performance of approximation algorithm

Figure 4 compares system throughput rates from the simulation to those from the approximation algorithm. *Figure 5* plots the relative error, which measures the accuracy of the approximation algorithm compared to the simulation model. The relative error can be defined as follows:

$$\varepsilon(n,B)=\left[TH_{\text{app}}(n,B)-TH_{\text{sim}}(n,B)\right]\times 100\% \big/ TH_{\text{sim}}(n,B) \quad (20)$$

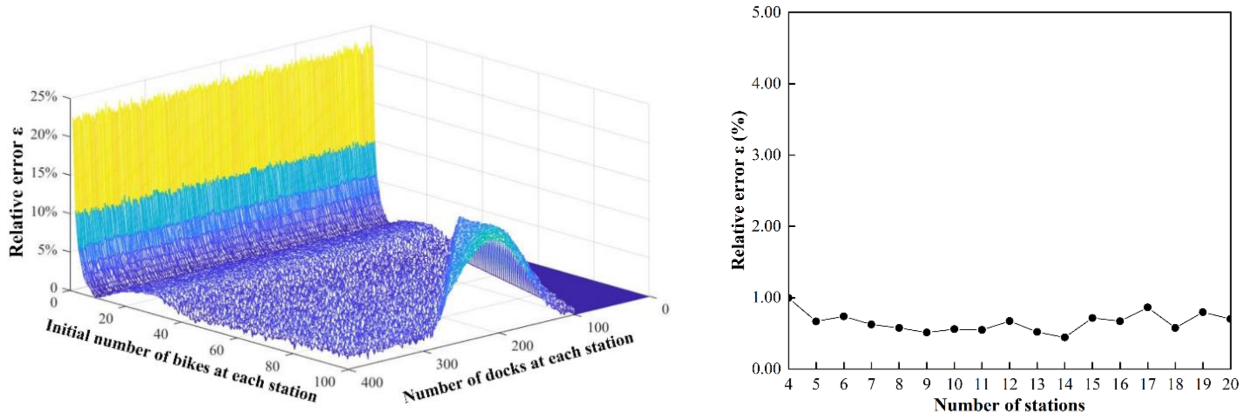
where $TH_{\text{app}}(n,\ B)$ and $TH_{\text{sim}}(n,\ B)$ denote the system throughput rates from the approximation algorithm and from the simulation model, respectively. *Figure 5a* shows that the relative error decreases from 23% to 2% as the fleet size and total number of docks increase. *Figure 5a* illustrates that the relative error decreases from 23% to 2% as the fleet size and total number of docks increase. However, the throughput rate experiences a peak of less 5% error rates as the number of docks increases from approximately 100 to 250, which can be attributed to the significant decrease in throughput rate for scenarios with 100 to 250 docks, as shown in *Figure 4*. Such accuracy performance implies that the approximation algorithm is more advantageous for its applications on large-scale FDBSS.



a) Simulation *b) Approximation*

Figure 4 – BSS effective throughput rates vs. the fleet size and docks

The relative errors of the approximation algorithm are also tested for different numbers of stations. We set the number of stations from 4 to 20 stations, respectively, and accordingly updated the parameters, including τ, λ, P and η. *Figure 5b* demonstrates that the relative errors of the approximation algorithm are bounded by less than 1%. The accuracy of the algorithm appears to be independent of the number of stations. Practically, our approximation algorithm is promising for industrial applications due to its ease of programming and its ability to provide accurate evaluations of throughputs in large-scale FDBSSs.



a) *b)*

Figure 5 – Relative error of effective throughput rates between simulation and approximation

Effects of bike fleet size on throughput rate

Figure 6 illustrates the changes in nominal and effective system throughput rates with varying bike fleet sizes for different dock allocations. The nominal and effective throughput rates are equal for fleet sizes below their optimal values: 231 bikes (point B) for 100×4 docks and 530 bikes (point C) for 400×4 docks. These rates remain constant for fleet sizes ranging from 170 (point A) to 231 (point B) for 100×4 docks and from 170 (point A) to 530 (point C) for 400×4 docks. However, when the fleet size exceeds these optimal values, the effective throughput rates remain constant and more users are unable to return bikes, thereby reducing the FDBSS service quality. The effective throughput rate increases up to an upper bound as fleet size increases, confirming **Property 1**. Differentiating between effective and nominal throughputs clarifies the misconception that larger bike fleet sizes always enhance FDBSS efficiency. This distinction is crucial for determining the appropriate bike fleet size.

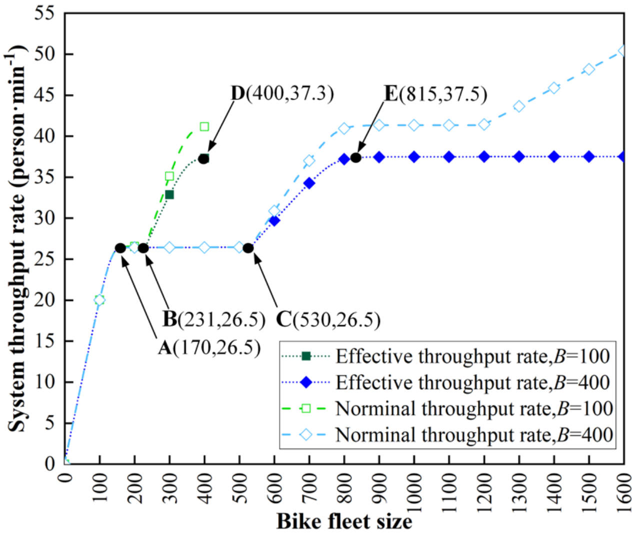
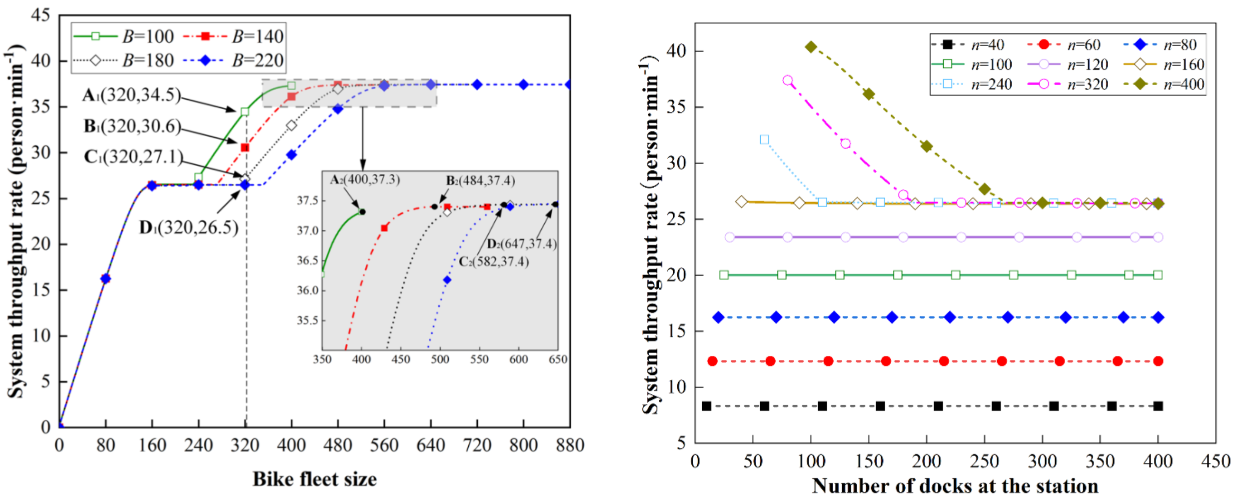


Figure 6 – Nominal and effective throughput rates vs. bike fleet sizes

Effects of number of docks on throughput rate

Figure 7a shows the effective system throughput rates with the bike fleet size for 4×100, 4×140 and 4×180 and 4×220 docks. In the figure, if fleet size is less than 224 bikes or more than 648 bikes, the effective system throughput rates are identical for the four dock allocations; if fleet size is between 224 and 648 bikes, say, 320 bikes, refer to points A_1, B_1, C_1 and D_1, the effective system throughput rates are 34.5, 30.6, 27.1 and 26.5 pax/min for 4×100, 4×140 and 4×180 and 4×220 docks, respectively. This implies that allocating more docks can decrease effective system throughput rates for certain fleet sizes. *Figure 7a* also displays that if the fleet sizes exceed their optimal values, 400, 484, 582 and 647 bikes, associated with points A_2, B_2, C_2 and D_2, for the four dock allocations, the effective throughput rate remains nearly constant as docks increase.

Figure 7b illustrates the relationship between the FDBSS throughput rate and the number of docks at each station for varying bike fleet sizes. For small fleet sizes (n =40-160), increasing the number of docks does not enhance the effective system throughput rate. For large fleet sizes (n =240–400), it initially decreases the throughput rate before stabilising. This occurs because small fleet sizes have low probabilities of fully occupied docks and high probabilities of successful bike returns, meaning additional docks do not significantly increase the throughput rate. Conversely, with large fleet sizes, more docks can lead to bike accumulation at certain stations, reducing bike availability at others and thereby decreasing the system throughput rate. Overall, observations from *Figure 7* indicate that increasing the number of docks does not improve the effective system throughput rate for small and large fleet sizes and can even reduce it for medium fleet sizes. Thus, the belief that allocating more docks at stations improves the FDBSS efficiency is a misconception.



a) bike fleet size *b) number of docks*

Figure 7 – Effects on effective throughput rate

Diagnosis of station states

This section aims to demonstrate whether Proposition 2 can diagnose stations with bike surplus and deficiency. *Table 2* lists the parameter settings for four scenarios, where scenarios 1, 3 and 4 have identical average user arrival rates. In addition, according to Jiang [29], a well-performing FDBSS should ensure that each station has at least 20% of its total docks vacant and 20% of its total docks occupied during peak periods. Thus, we set $\varphi_1 = 0.2$, $\varphi_2 = 0.8$, $\omega_1 = \omega_2 = 0.8$.

Table 2 – Scenarios settings

	Scenario 1	Scenario 2	Scenario 3	Scenario 4
Initial number of bikes in stations	$(70,70,70,70)$	$(70,70,70,70)$	$(70,70,70,70)$	$(70,70,70,70)$
Number of docks	$(100,100,100,100)$	$(100,100,100,100)$	$(100,100,70,130)$	$(100,100,100,100)$
Average user arrival rate (person/minute)	$(10,10,10,10)$	$(6,6,10,10)$	$(10,10,10,10)$	$(10,10,10,10)$
Routing matrix	$\begin{pmatrix} 1/7 & 2/7 & 3/7 & 1/7 \\ 1/7 & 2/7 & 3/7 & 1/7 \\ 3/11 & 2/11 & 4/11 & 2/11 \\ 5/11 & 1/11 & 3/11 & 2/11 \end{pmatrix}$	$\begin{pmatrix} 1/7 & 2/7 & 3/7 & 1/7 \\ 1/7 & 2/7 & 3/7 & 1/7 \\ 3/11 & 2/11 & 4/11 & 2/11 \\ 5/11 & 1/11 & 3/11 & 2/11 \end{pmatrix}$	$\begin{pmatrix} 1/7 & 2/7 & 3/7 & 1/7 \\ 1/7 & 2/7 & 3/7 & 1/7 \\ 3/11 & 2/11 & 4/11 & 2/11 \\ 5/11 & 1/11 & 3/11 & 2/11 \end{pmatrix}$	$\begin{pmatrix} 1/6 & 1/3 & 1/3 & 1/6 \\ 1/3 & 1/6 & 1/6 & 1/3 \\ 1/3 & 1/6 & 1/6 & 1/3 \\ 1/6 & 1/3 & 1/3 & 1/6 \end{pmatrix}$

Table 3 lists the simulated and approximated probabilities of bike surplus and deficiency for the four scenarios. It shows that the approximated and simulated probabilities are consistent, which suggests that Proposition 3 can effectively diagnose. *Figure 8* displays the nominal and effective system throughput rates for the four scenarios. It indicates that bike surplus and deficiency occur in scenarios 1, 2 and 3 (*Figure 8a–c*), not in scenario 4 (*Figure. 8d*). Only in scenario 4 can the condition $(\mathbf{I}-\mathbf{P}^{\mathrm{T}})\boldsymbol{\lambda}=\mathbf{0}$ be verified to hold. The FDBSS then has equal and maximal nominal and effective throughput rates, as shown in *Figure 8d* and is thus completely self-balanced.

Table 3 – Comparison of approximations and simulation probabilities of bike surplus and deficiency

		Approximation				Simulation			
Station No.		1	2	3	4	1	2	3	4
Scenario 1	$P(X_i < 20)$	0.9923	0.9998	0.0000	0.9997	0.9735	0.9955	0.0000	0.9625
	$P(X_i > 80)$	0.0000	0.0000	0.9521	0.0000	0.0000	0.0000	0.9885	0.0000
Scenario 2	$P(X_i < 20)$	0.0109	0.4210	0.3906	1.0000	0.0000	0.3775	0.2820	0.9976
	$P(X_i > 80)$	0.6379	0.0589	0.0707	0.0000	0.7085	0.0000	0.0360	0.0000
Scenario 3	$P(X_i < 20)$	0.9178	0.9988	0.0000	0.9898	0.8976	0.9905	0.0000	0.8776
	$P(X_i > 80)$	0.0000	0.0000	0.9676	0.0000	0.0000	0.0000	0.9895	0.0000
Scenario 4	$P(X_i < 20)$	0.4750	0.4750	0.4750	0.4750	0.4420	0.5145	0.4860	0.4630
	$P(X_i > 80)$	0.0415	0.0415	0.0415	0.0415	0.0005	0.0025	0.0000	0.0275

Note: The colours of the numbers indicate the stations with bike deficiency, surplus and balance, respectively.

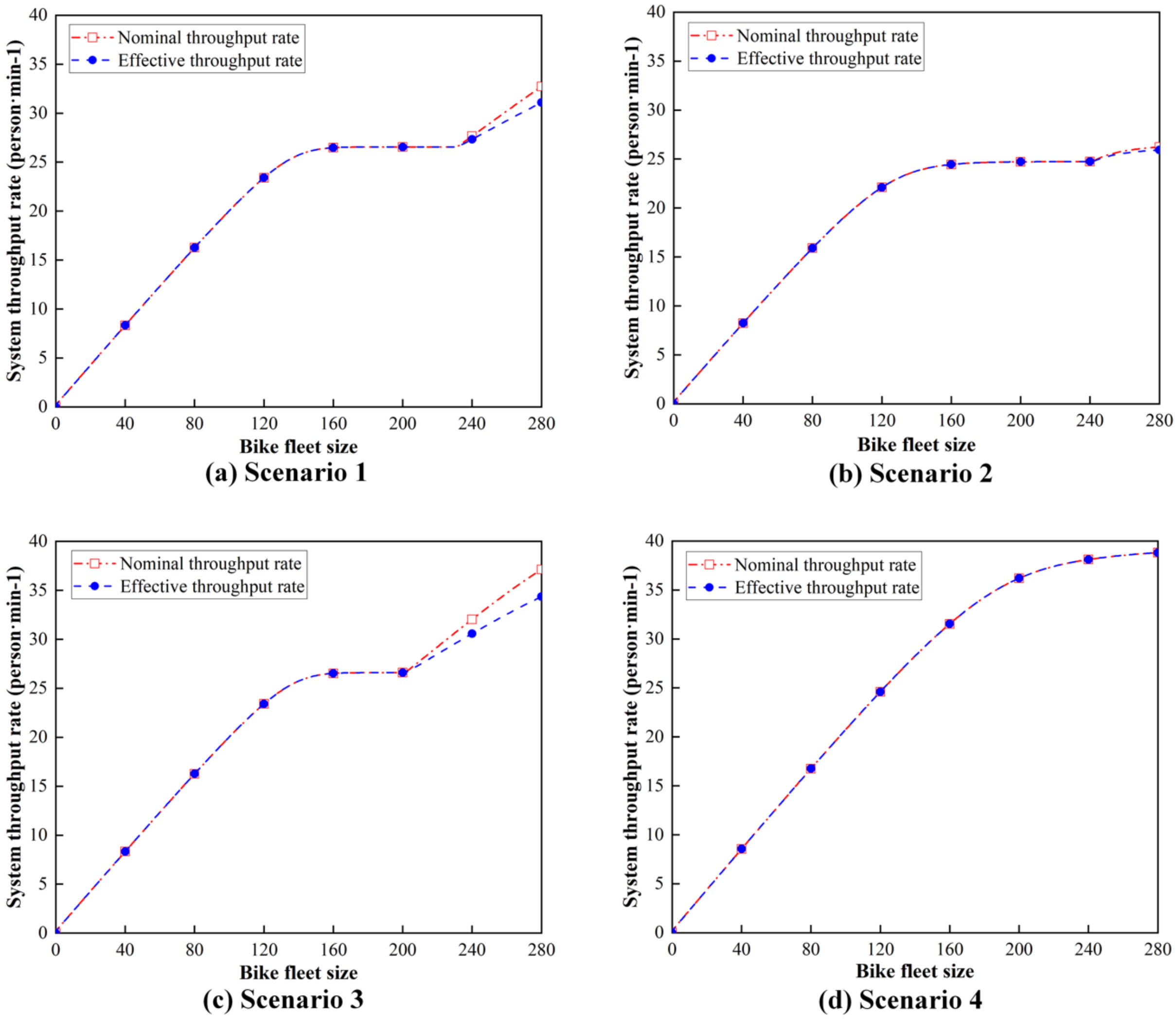


Figure 8 – Throughput rate vs. bike fleet size

4.2 Experiments on a real-world BSS

The proposed model can be applied to real-world FDBSS. We implemented it in the Higher Education Town, Dushu Lake, Suzhou, China, as illustrated in *Figure 9*. This system comprises 37 stations, 442 bikes and 1260 docks. Operational data was collected over 24 hours on 12 October 2019, including user rental and return times, as well as station, dock and user identities. Data analysis yielded the average travel time, routing matrix and an average effective system throughput rate of 86.58 pax/h. We set $\varphi_1 = 0.2$, $\varphi_2 = 0.8$, $\omega_1 = \omega_2 = 0.8$. To estimate user arrival rates at stations, we used bike operation data and video footage from cameras at stations to identify lost users who were unable to rent bikes. We also identified users who could not return bikes at fully occupied stations and redirected them to available stations based on their features, bodily form and clothing. Subsequently, we established the overflow routing matrix. Observations indicate that users generally return bikes to the nearest station from their destination stations. For brevity, the average cycle times, adjacency return and routing matrices, and dock allocation are not listed here.

By using the approximation algorithm, we can easily compute that the effective system throughput rate is 88.4 pax/h, with a relative error of 2.04% compared to the real throughput rate of 86.58 pax/h. *Figure 10* displays the average user arrival rates and effective station throughput rates. The FDBSS has 30 bike-deficient stations and four bike-surplus stations (No. 4, 5, 7 and 34), where user demands at stations 1, 9, 22, 26, 27 and 37, exceeding more than 5 pax/h, cannot rent bikes. The bike queues within the FDBSS are highly unbalanced. Relying solely on users is insufficient to achieve self-balance, leading to potential unavailability of bikes for renting and returning at surplus and deficient stations.

Figure 9 – Bike station locations in case example

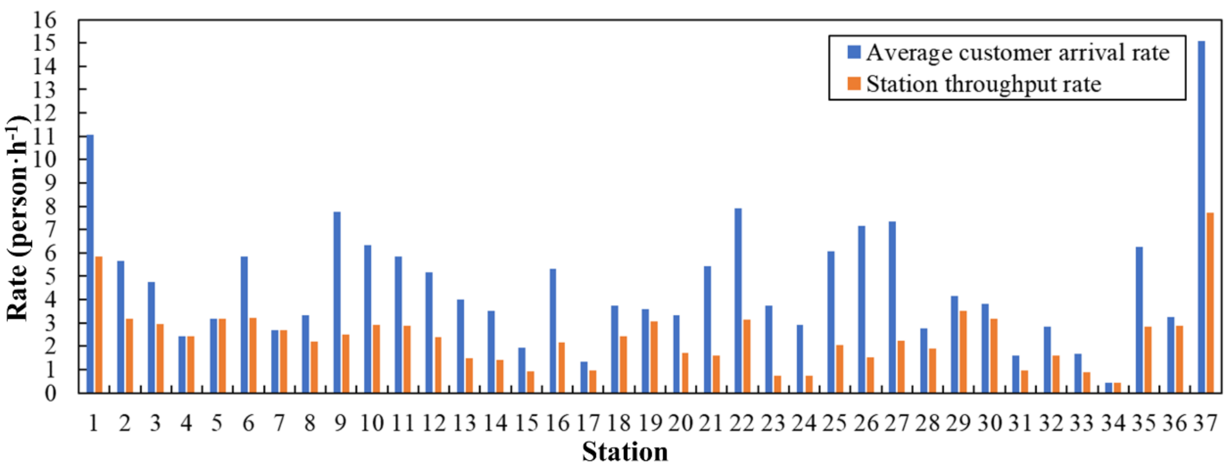


Figure 10 – Average user arrival rates and effective throughput rates

Figure 11 shows the effective system throughput rates with bike fleet size. As the fleet size expands from 0 to 225 bikes, the effective system throughput rate gradually increases in a zigzag pattern, reaching its maximum value of 88.4 pax/h once the fleet size exceeds the optimal 225 bikes. The fleet size of 442 bikes is clearly excessive.

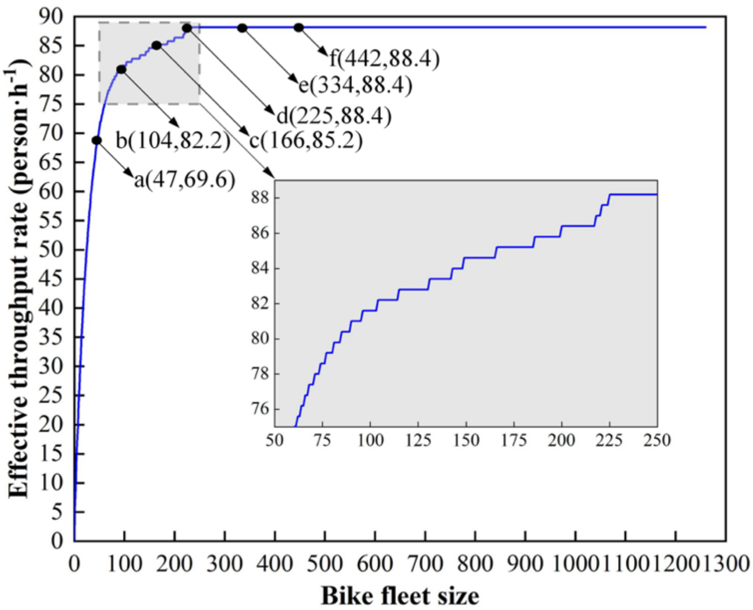


Figure 11 – Effective throughput rates vs. bike fleet sizes

5. CONCLUSION

This paper presents a closed queuing network model to approximate bike queues in finite docked bike sharing systems (FDBSS), accounting for bike spillovers from stations. The derived approximation algorithm can efficiently solve for system throughput rates, expected bike dwell time and optimal bike fleet size. Additionally, a method for diagnosing stations with bike surplus and deficiency is developed, revealing operational properties of the FDBSS. The proposed algorithm and diagnostic method are straightforward to code and offer high solution efficiency, indicating strong potential for industrial applicability. Furthermore, our study dispels the misconception that simply expanding bike fleet size and adding more docks will invariably enhance the FDBSS efficiency.

Future research can be explored in several directions. First, this paper assumes stationary user arrivals, whereas time-varying user arrivals are common in the real FDBSS. Thus, the approximation algorithm could be extended to examine the FDBSS with dynamic user arrival rates. Second, FDBSS planning involves various decision variables, including station locations, spacing and cycling distances, which are not investigated in this study and should be considered in future research. Third, we have proven that if the condition $(\mathbf{I}-\mathbf{P}^{\mathrm{T}})\boldsymbol{\lambda}=\mathbf{0}$ holds, the FDBSS can maximize its throughput rate. It is necessary to optimize dock allocation to adjust user flows spilling over from stations to meet this condition, allowing the FDBSS to operate at maximum system throughput without the need for bike relocation.

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有限车桩自行车共享系统中排队近似

摘要：
面向有限车桩共享自行车系统，本文提出了近似估计自行车排队的封闭排队网络模型。该模型能考虑因站点满桩和空桩而导致的自行车溢出和用户流失，确定系统吞吐率及其他绩效指标。为了克服封闭排队网络的计算困难，基于所建模型提出了一种近似算法。通过揭示有限车桩共享自行车系统的固有特性表明: (i) 系统有效吞吐率随自行车数量增加而增加，最终收敛到上限值。(ii) 增加车桩不一定增加甚至会降低有效吞吐率。(iii) 系统在某些条件下能实现自平衡，此时各站点的自行车不会过剩或不足，且系统吞吐率最大。(iv) 用户还车路线上只要存在站点的空桩概率大于零，用户有限次尝试归还即能成功还车。通过小算例和真实案例证明了近似计算法的准确性和适应性以及系统的特性。

关键词：
有限车桩自行车共享系统、吞吐率、近似算法、封闭排队网络